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Lectures on Abelian Lagrangian Algebraic Geometry

In these lectures we give a detailed explanation of algebraic geometric approach to Lagrangian geometry developed by Andrei Tyurin in his last works. A brief digest of this theory was given in my talk on the previous conference in Tokyo.

In the first lecture we recall the basic technical ingredients of ALAG. Namely, let (M, ω) be smooth real $2n$-dimensional symplectic manifold equipped with a smooth hermitian line bundle L with hermitian connection a such that $c_1(L) = m[\omega]$ in $H^2(M, \mathbb{Z})$ for some $m \in \mathbb{Z}$ and $c_1(L) \equiv c_1(K) \pmod 2$, where $c_1(K)$ is the canonical class of M. In this purely real setup one can build an infinite dimensional complex Kähler variety $\mathfrak{P}^{\mathrm{hw}}$, of *half weighted Planck cycles*, which is a principal U(1)-bundle over the variety $\mathfrak{B}^{\mathrm{hw}}$, of *half weighted Bohr – Sommerfeld cycles*. The volume form induced by a half weight of Planck cycle provides us with the moment map for the fiberwise U(1)-action and the symplectic quotient $\mathfrak{P}^{\mathrm{hw}}//\mathrm{U}(1) = \mathfrak{B}^{\mathrm{hw}}_t$ is another infinite dimensional complex Kähler variety — the space of half weighted Bohr–Sommerfeld cycles of the fixed volume t. The construction of the Kähler manifolds in question is equivariant w.r.t. the symplectomorphisms of M. Thus, the Poisson algebra of M is naturally represented in the Poisson algebra of $\mathfrak{B}^{\mathrm{hw}}_t$. In some sense, this is a non-linear version of the Dirac quantization concept.

The most important part of the theory is the Bortwick–Paul–Uribe map associated with an integrable complex structure I on M compatible with ω. In a presence of such a Kähler structure (I, ω, g), L turns to a holomorphic line bundle over M and produces a series of 'conformal block spaces' $H_{(k)} = H^0(M, L^{\otimes k})$. The BPU-map $\beta : \mathfrak{P}^{\mathrm{hw}} \to H^*_{(k)}$ takes a half-weighted Planck cycle to a complex linear form on the space of conformal blocks. It is holomorphic and the dual to its differential is induced by restriction map, which takes a holomorphic section of L to its restriction onto the Planck cycle.

In the second lecture we consider an application of the previous technique in the case when (M, ω) is a completely integrable dynamical system, i.e. M admits a Lagrangian toric fibration $\pi : M \to \Delta$ over some polytope. It turns out that such a fibration contains a finite set of Bohr–Sommerfeld cycles. Using the Gauss map and Maslov classes, all these fibers can be equipped with half weights. Moreover, for any integrable Kähler structure (I, ω, g) on M the BPU-map takes these fibers to distinguished vectors $\varphi_i \in H^*_{(k)}$, which can be putted into $H_{(k)}$ by means of hermitian form on $L^{\otimes k}$. For a fixed real polarization π we get a collection of sections φ_i of conformal blocks bundle over the moduli space of polarized Kähler structures on M. Vice versa, for a fixed Kähler polarization, each real integrable structure can be equipped with a collection of vectors in a fixed conformal blocks space. Since real integrable structures typically form a discrete system like a graph, we get a kind of discrete field theory formed by transition matrices between the vectors φ_i along the edges of this graph.

Thus, we get some generalization of Knizhnik–Zamolodchikov theory. The simplest example of interaction between real and complex polarizations provided by ALAG and BPU is

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the classical theory of theta-functions, when $M = \mathbb{C}^n/\Lambda$ is an abelian variety. Besides this example, we are going to discuss the non-abelian theta functions that appear when M is the moduli space of holomorphic rank 2 bundles on a Riemann surface Σ. This example is also quite known: Kähler structures on M are induced by complex structures on Σ, the real polarizations on M are provided by 'paints decompositions' of Σ, which are organized into the 'graph of paints'.

In the third lecture we are going to discuss two extensions of ALAG approach. The first is some 'complexification' of Bohr–Sommerfeld conditions developed in one of the most designing Tyurin's preprints[2] We can vary the initial U(1)-connection on L in a pencil of $\mathbb{C}^*$-connections a_λ and modify the Bohr – Sommerfeld condition on S by requesting that L_S should admit a horizontal trivialization w.r.t. a_λ. Then the above story becomes fibered over $\mathbb{P}_1$. In particular, the BS_λ-fibers of a given real polarization will form a local system over the line where λ runs through, or equivalently, draws some Riemann surface ramified over this line.

If the time allows we also will discuss the ALAG-patterns in the most resent approaches to quantization based on homotopy algebra. It is quite popular today to think of correlators in quantum field theories as higher operations in appropriate strong homotopy algebra (A_∞, E_∞, e.t.c.). The BV-formalism used by physicists to speak about these higher operations looks like 'lagrangian reformulation' of the ordinary homological algebra, but these reformulation leads to several purely non-commutative generalizations of the ALAG ideology.

[2] *A. Tyurin. Complexification of Bohr-Sommerfeld conditions.* `arXiv: math.AG/9909094`.